Financial Economics

Basics

Financial assets can be categorized as follows:

- **Riskless assets:** assets whose future values can be known with certainty. These are purely theoretical, however some items are considered as (almost) riskless assets, e.g. government bonds and savings accounts.
- **Risky assets:** assets whose future values may depend on the realization of some events (e.g. health insurance, stocks, options, ...)
- **Derivatives:** risky assets whose future values depend on the price of other assets.

A **zero-coupon bond** with maturity t yields no payment before period t and pays one in period t. We denote by $p_{0,t}$ the price of a zero-coupon bond with maturity t at time zero. In other words, $p_{0,t}$ is the value at time zero of one unit at time t. In absence of arbitrage, $p_{0,0} = 1$.

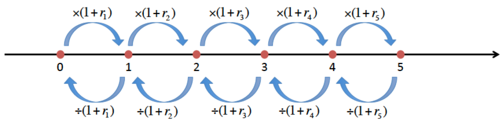
The concept of $p_{0,t}$ allows us to define r_t, the discrete per period forward interest rate between timestamps $t-1$ and t, by:

$$p_{0,t} = \frac{p_{0,t-1}}{(1+r_t)}$$

which implies

$$p_{0,t} = \frac{1}{\prod_{i=1}^{t}(1+r_i)}$$

The concepts of compounding and discounting (see next figure), naturally follow.



More formally, we have:

- **Discounting:** process of determining present value of future monetary amount x_t.
- **Compounding:** process of determining the current value at time t (in the future) of a monetary amount x_0.

It holds:

$$PV(x_t) = \frac{x_t}{\prod_{i=1}^{t}(1+r_i)}$$

$$CV_t(x_0) = \prod_{i=1}^{t}(1+r_i)x_0$$

The **spot interest rate** $\bar{r}_t$ for a zero-coupon bond with maturity t is defined as a geometric average of the per period forward rate of interest. In formulae:

$$p_{0,t} = \frac{p_{0,0}}{(1+\bar{r}_t)^t}$$

$$1+\bar{r}_t = \left[\prod_{i=1}^{t}(1+r_i)\right]^{\frac{1}{t}}$$

Another interesting concept is the yield to maturity, defined as the unique rate y such that, for a bond with price p_0 at time 0 which yields a series of payment a_t and a final payment of value Par it holds:

$$p_0 = \sum_{t=1}^{T}\frac{a_t}{(1+y)^t} + \frac{Par}{(1+y)^T}$$

Options and Financial Strategies

Derivatives are assets whose values mechanically depend on the value of other financial assets (the underlying). There are two macro-categories:

- **Forward contracts:** obligation to purchase (long position) or sell (short position) the underlying at a specified future price at a specified delivery date. Forwards are entered at no cost.
- **Option contracts:** right to purchase or sell a specified amount of the underlying at a specified exercise price at or before a specified expiration date. Options offer an advantage: the transaction does not occur if it is not profitable to the owner (the one who is in a long position). Of course, this advantage comes at a price, i.e. the long position pays an initial price to get the option. Selling an option implies an obligation, and you receive a compensation for this.

We first take a deeper look into **forward contracts**. They can help shaping risk exposure, for example with:

- **Hedging:** to insure against market price volatility (e.g. you have a product that you will sell in a year, but you fix the price now, as this allows to reduce potential losses, of course at the price of not earning more if the value of the underlying increases to more that the fixed price).
- **Speculation:** to exploit market price volatility.

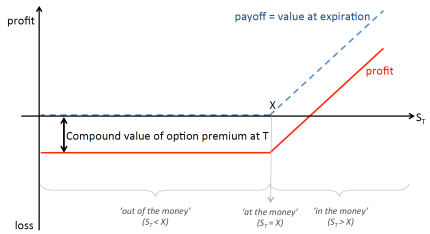
More formally, given a delivery price X and a future value S_T at time T, the long position gets $S_T - X$, while the short position gets $X - S_T$. This means, that if the underlying now has a value of S_t and you expect that it has value S_T^c at time T you should do the following:

- If $S_T^c > S_t$ you should aim to be in the long position of a forward contract with $X < S_T^c$ (or in a short position with $X > S_T^c$).
- If $S_T^c < S_t$ you should aim to be in a short position of a forward contract with $X > S_T^c$ (or in a long position with $X < S_T^c$).

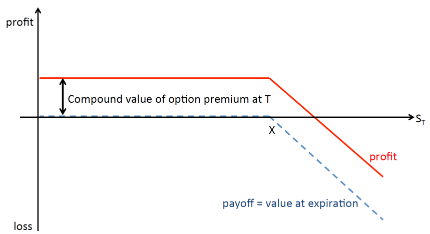
We now describe **options**. An option is a right to purchase/ sell a certain amount of the underlying at (European option) or before (American option) the expiration date T at exercise price X. In general, the long position is the option holder (has the right to exercise the option and pays a price for this), and the short position is the option writer (which has an obligation to facilitate the option's exercise and gets a compensation for this service). We distinguish two types of options.
The **call** (long position has the right to purchase an assrt for X at time T) works as follows: the long position pays a premium C_0 at time 0 and at time T it does the

following: if $S_T > X$, it gets a payoff of $S_T - X$ (and a profit of $S_T - X - CV_T(C_0)$), otherwise it gets a payoff of 0 (it does not worth it to pay X to get $S_T < X$) and the final profit is $-CV_T(C_0)$. On the other hand the short position always gets the premium C_0 and it has a payoff of 0 if the long position does not exercise the option (i.e. when $X > S_T$) and a payoff of $X - S_T$ if the long position exercises the option (i.e. when $X < S_T$). The following figure illustrate the scenario.

Profit-Loss Diagram Call Option: **Long Call**

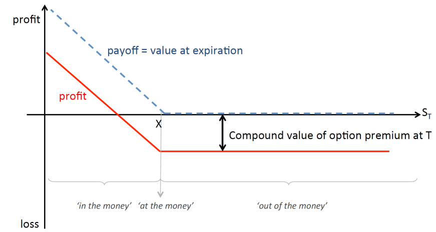


Profit-Loss Diagram Call Option: **Short Call**

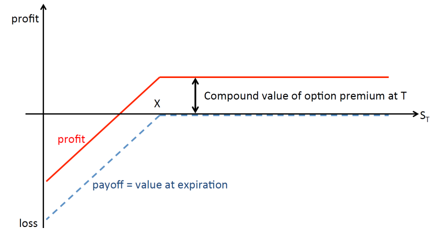


On the other hand, the **put** (where the long position has the right to sell an asset for X at time T) works as follows: the long position pays a premium P_0 at time 0 and at time T it does the following: if $S_T < X$ it sells the underlying and get a payoff of $X - S_T$ (which means a profit of $X - S_T - CV_T(P_0)$), otherwise the payoff is zero and the profit $-CV_T(P_0)$. The short position always gets the premium P_0 and the payoff depends on the choice of the long position. If the long position exercises the option, the payoff is $S_T - X$, otherwise it is zero. To get the profit, we add $CV_T(P_0)$ to the payoff. The situation is illustrated next.

Profit-Loss Diagram Put Option: **Long Put**

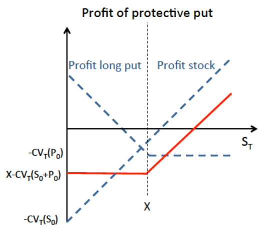


Profit-Loss Diagram Put Option: **Short Put**



Now that we have presented different options, we will explore some possible strategies to get a positive profit. First, note that if we have some expectation about the price development of the underlying we can use either **bullish strategies** (where we generate a profit if the underlying's price increases, e.g. long call and short put) or **bearish strategies** (where we generate a profit if the underlying's price decreases, e.g. long put and short call). In contrast, if we have no expectation about price deelopments but we have some ideas regarding the volatility of the underlying, we can use **non-directional strategies**, where we generate a profit depending on the underlying's actual volatility. Let's see some examples.

Protective put: if you want to insure against potential losses due to decline in stock value, you can buy a long put on the sock. This strategy guarantees a capped loss at the price of a lower gain if the stock price increases.

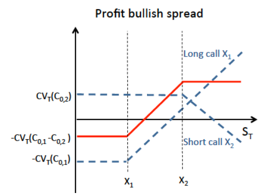


Payoffs and profits in T:

		$S_T \leq X$	$S_T > X$
Payoff	stock	S_T	S_T
	+ put	$X - S_T$	0
	= total	X	S_T
Profit		$X - C^P$	$S_T - C^P$

with $C^P = CV_T(S_0 + P_0)$

Spread I: wish to take advantage of higher future price of underlying. Of course you could buy the stock, long a call or short a put. An alternative is to construct a portfolio that allows to reduce the risk (i.e. if your prediction is wrong, you don't loose too much) at the cost of giving up part of the profits. A possible bull spread strategy is buying a long call with strike X_1 and premium $C_{0,1}$ and a short call with strike $X_2 > X_1$ with premium $C_{0,2}$. Te results are shown below.

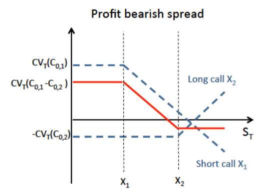


Payoffs and profits in T:

		$S_T \leq X_1$	$X_1 < S_T \leq X_2$	$S_T > X_2$
Payoff	+ call X_1	0	$S_T - X_1$	$S_T - X_1$
	- call X_2	0	0	$-S_T + X_2$
	= total	0	$S_T - X_1$	$X_2 - X_1$
Profit		$-C^P$	$S_T - X_1 - C^P$	$X_2 - X_1 - C^P$

with $C^P = CV_T(C_{0,1} - C_{0,2}) > 0$

Spread II: alternatively, if you wish to take advantage of lower future price of the underlying, you could sell the stock, short a call, or buy a put. Another opportunity is bearing a spread strategy, which allows to hedge against the risk that we have wrong expectation on future price of the underlying. An example could be shorting a call with strike X_1 and premium $C_{0,1}$ and buying a long call with strike $X_2 > X_1$ and premium $C_{0,2}$. The situation is shown below.

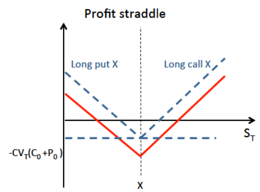


Payoffs and profits in T:

		$S_T \leq X_1$	$X_1 < S_T \leq X_2$	$S_T > X_2$
Payoff	- call X_1	0	$-S_T + X_1$	$-S_T + X_1$
	+ call X_2	0	0	$S_T - X_2$
	= total	0	$-S_T + X_1$	$X_1 - X_2$
Profit		$-C^P$	$-S_T + X_1 - C^P$	$X_1 - X_2 - C^P$

with $C^P = CV_T(C_{0,2} - C_{0,1}) < 0$

Straddle: useful when a large move in stock's price is expected, but there is uncertainty about the direction. In this case, we buy a long call with strike X together with a long put with strike X. The result is shown below.



Payoffs and profits in T:

		$S_T \leq X$	$S_T > X$
Payoff	Call	0	$S_T - X$
	+ Put	$X - S_T$	0
	= total	$X - S_T$	$S_T - X$
Profit		$X - S_T - C^P$	$S_T - X - C^P$

with $C^P = CV_T(C_0 + P_0)$

- Think the movement more likely bullish? $\Longrightarrow$ buy 2 calls (**strap**)
- Think the movement more likely bearish? $\Longrightarrow$ buy 2 puts (**strip**)

Option Valuation

An arbitrage opportunity is a financial strategy that yields a sure cash-flow $A_0 \geq 0$ at time 0 and cash-flow $A_T \geq 0$ almost surely at time T and there is at least one state of the world where one inequality is strict.

- $A_0 > 0$ and $A_T \geq 0$ almost surely is an arbitrage.
- $A_0 \geq 0$ and $A_T \geq 0$ almost surely with $Pr[A_T > 0] > 0$ is an arbitrage.

If we assume that the market clears out arbitrage opportunities, we can derive a number of interesting results.

(a) $\max[S_0 - PV(X), 0] \leq C_0 \leq S_0$

(b) $\max[PV(X) - S_0, 0] \leq P_0 \leq PV(X)$

(c) $S_0 + P_0 = \frac{X}{(1+r_f)^T} + C_0$

(d) **Two-State Option Valuation Model:** a stock that sells at S_0 and has either value S_1^u or S_1^d with $S_1^u > S_1^d$, a call option with strike $X \in (S_1^d, S_1^u)$, and a risk-free asset with yearly interest rate r_f. In absence of arbitrage $C_0 = \alpha S_0 + \beta$, with

$$\alpha = \frac{S_1^u - X}{S_1^u - S_1^d} \qquad \beta = \frac{(S_1^u - X)S_1^d}{(1+r_f)(S_1^d - S_1^u)}$$

(a) **Upper bound:** one portfolio costs C_0 and returns $\max[S_T - X, 0]$, the other one costs S_0 and returns S_T. Assume $C_0 > S_0$. Then do the following: sell a call and buy a stock. We have: $A_0 = C_0 - S_0 > 0$ and $A_T = S_T - \max[S_T - X, 0] = \min[X, S_T] > 0$. Contradiction. **Lower bound:** one portfolio costs C_0 and returns $\max[S_T - X, 0]$, another one costs S_0 and returns S_T, and the third one costs $PV(X)$ and return X. Assume $C_0 < S_0 - PV(X)$. Strategy: sell S_0 and buy $C_0 + PV(X)$. We have: $A_0 = S_0 - PV(X) - C_0 > 0$, $A_T = X + \max[S_T - X, 0] - S_T = \max[0, X - S_T] \geq 0$. Contradiction. $C_0 \geq 0$ is obvious.

(b) **Upper bound:** one portfolio costs P_0 and returns $\max[X - S_T, 0]$, the other one costs $PV(X)$ and returns X. Assume $P_0 > PV(X)$. Strategy: buy $PV(X)$ and sell P_0. We have: $A_0 = P_0 - PV(X) > 0$, $A_T = X - \max[X - S_T, 0] = \min[S_T, X] \geq 0$. Contradiction. **Lower bound:** one portfolio costs P_0 and returns $\max[X - S_T, 0]$, the second one costs S_0 and returns S_T, the third one costs $PV(X)$ and returns X. Assume $P_0 < PV(X) - S_0$. Strategy: buy the stock and put, sell $PV(X)$. It holds: $A_0 = PV(X) - S_0 - P_0 > 0$, $A_T = S_T + \max[X - S_T, 0] - X = \max[0, S_T - X] > 0$. Contradiction. $P_0 > 0$ is obvious.

(c) Similar as before, consider the put, call, stock, and risk-free asset opportunities. Assume first $S_0 + P_0 < \frac{X}{(1+r_f)^T} + C_0$ and find an opportunity, then do the same for $S_0 + P_0 > \frac{X}{(1+r_f)^T} + C_0$. By contradiction you get the result.

Option valuation depends on multiple factors, as summarised in the following table.

	Price of a Call option	Price of a Put option
Underlying stock price	+	−
Exercise price (strike)	−	+
Volatility of underlying stock price	+	+
Time to expiration	+	+/−
Interest rate	+	−

Finally, a fundamental result in option valuation is the Black-Scholes Formula. We assume that the stock at time t denoted $S(t)$ varies as follows:

$$dS(t) = \mu S(t) + \sigma S(t) dB(t)$$

where $B(t)$ is a standard Brownian motion. It holds:

$$C_0 = S_0 N(d_1) - X e^{-rT} N(d_2)$$

with $d_1 = \frac{\ln(\frac{S_0}{X} + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

Pricing by Arbitrage

The setting for this chapter is the following. Given two periods ($t = 0, 1$), a space Ω that represents the contingent states at time $t = 1 : \omega \in \{1, \ldots, \Omega\}$ and K assets $k \in \{1, \ldots, K\}$ available at time $t = 0$, we define:

- a_k^ω is the value of asset k at time $t = 1$ in state ω.
- p_k is the price of asset k at $t = 0$.
- A portfolio (z_k) is a vector of asset quantities. z_k denoted the quantity of asset k held in the portfolio.
- A market is the data (a_k^ω, p_k).
- The price of the portfolio is $\sum_k p_k z_k$.
- The value of the portfolio in state ω is $\sum_k z_k a_k^\omega$.

If an asset k have values a_k^ω independent of ω, we call if a risk-free asset. We also say that an asset k can be replicated by a subset of assets, indexed by $i \in S$, if there exist constants $(z_i)_{i \in S}$ such that $a_k^\omega = \sum_{i \in S} a_i^\omega z_i \forall \omega$. In such a case, the asset k is said to be redundant. We now look at the concept of complete market.

- **Definition:** a market (a_k^ω, p_k) is complete if for any asset b^ω there exist constants (z_k) such that $b^\omega = \sum_k z_k a_k^\omega$.
- Market are complete when all revenue configurations are replicable through some portfolio.
- In complete markets it has to hold $\Omega \leq K$.
- A market is complete if and only if the payoff matrix a (with dimension $K \times \Omega$) has rank Ω.
- If a market is complete with $K > \Omega$ there are Ω independent assets. Thus, one can eliminate $K - \Omega$ redundant assets which are linear combinations of the Ω independent assets.

An arbitrage portfolio is a portfolio $(z_k)_{k\in 0,\dots,K}$ such that $\sum_k a_k^\omega z_k \geq 0 \forall\omega$ and $\sum_k p_k z_k \leq 0$, where at least one of these $\Omega + 1$ inequalities is strict.

We say that a market is arbitrage free if there is no arbitrage portfolio. A market (a_k^ω, p_k) without arbitrage opportunities satisfies the **Law of one price**: if for two assets i and j we have $a_i^\omega = a_j^\omega$ for all ω, then $p_i = p_j$. If that's not the case, w.l.o.g. because $p_i < p_j$ we would then have an arbitrage opportunity by selling one unit of asset j and buying one unit of asset i.
The No-Arbitrage Theorem: A market (a_k^ω, p_k) is arbitrage-free if there exist a vector q such that $p = Aq$ and every component of q is strictly positive. Note that they may exist several state-price vectors q.
Given a state-price vector $(q_\omega)_{\omega\in\Omega}$ we define the risk neutral probability $(\pi_\omega^*)_{\omega\in\Omega}$ with $\pi_\omega^* = \frac{q_\omega}{\sum_\omega q_\omega}$. We also define:

- $r_k^\omega = \frac{a_k^\omega}{p_k}$: the return of asset k in ω.
- Using $p_k = \sum_\omega a_k^\omega q_\omega$ we get

$$\mathbb{E}\left[r_k^\omega\right] = \sum_\omega \pi_\omega^* r_k^\omega = \frac{1}{\sum_\omega q_\omega}$$

Arrow-Debreu Securities: the Arrow-Debreu security a_{ω_0} is a vector of dimension $|\Omega|$ with a 1 in component ω_0 and zero otherwise. We can show two things:

- If the market is arbitrage free and q is a state-price vector, then the price of a_{ω_0} is q_{ω_0}.
- A market that contains all Arrow-Debreu securities is complete.

Assume an arbitrage free market with state-price vector q and corresponding risk neutral probabilities π^*. Assume that this market contains a risk free asset that pays 1 in all possible states of the world. We denote by $p_0 = \sum_\omega q_\omega$ its price and by $r_f = \frac{1}{p_0}$ its return. Then:

- For all assets k in the market we have $\mathbb{E}\left[r_k^\omega\right] = r_f$.
- If the market contains the Arrow-Debreu security a_{ω_0} its price is $q_{\omega_0} = \sum_\omega q_\omega \frac{q_{\omega_0}}{\sum_\omega q_\omega} = p_0 \pi_{\omega_0}^*$.

Given an arbitrage-free market $M = (a_k^\omega, p_k)$, define $\mathcal{Q}_M$ the set of state-price vectors for this market.

- $\mathcal{Q}_M$ is a convex set.
- $\mathcal{Q}_M$ is empty iff M offers arbitrage opportunities.
- $\mathcal{Q}_M$ is reduced to a point iff is complete and does not offer arbitrage opportunities.
- If $q, q' \in \mathcal{Q}_M$, then:

$$\sum_\omega \left(q_\omega - q'_\omega\right) a_k^\omega = 0 \forall k$$

For any claim $c = (c_1, \dots, c_\Omega)$ not necessarily replicable in the market M, the set of prices p_c for such a claim that would not generate arbitrage opportunities is:

$$\{p_c | p_c = \sum_\omega q_\omega c_\omega \text{ for some } q \in \mathcal{Q}_M\}$$

Moreover:

- This set of no-arbitrage prices is a singleton iff c is replicable in M.
- If c is not replicable in the market, then adding it to the market with a price within arbitrage bounds will reduce the set $\mathcal{Q}_M$. Its dimension will decrease by one.

Introduction to the Economic Analysis of Asset Markets

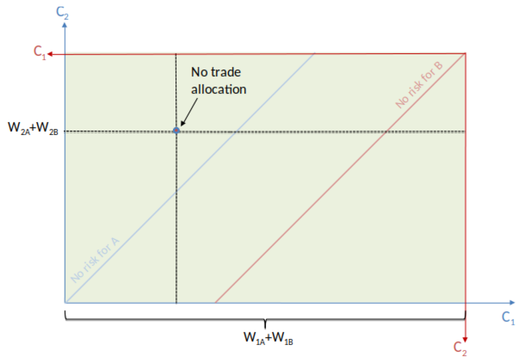
We start by a simple example with two agent A and B. Ex-post, there are two state of the world:

- In state $\omega = 1$, agents get income w_A^1 and w_B^1.
- In state $\omega = 2$, agents get income w_A^2 and w_B^2.

We denote with $c_A^1, c_A^2, c_B^1, c_B^2$ the consumptions of the agents in their respective state. The goal of the agent is getting a high consumption. Note, however, that since we assume that no value can be created out of nothing, it must hold that:

$$c_A^1 + c_B^1 = w_A^1 + w_B^1$$
$$c_A^2 + c_B^2 = w_A^2 + w_B^2$$

Of course, the two agents can decide to not trade. The no-trade situation corresponds to $c_i^1 = w_i^1$, $c_i^2 = w_i^2$ for $i \in \{A, B\}$. The situation can be represented graphically with an edgeworth box.



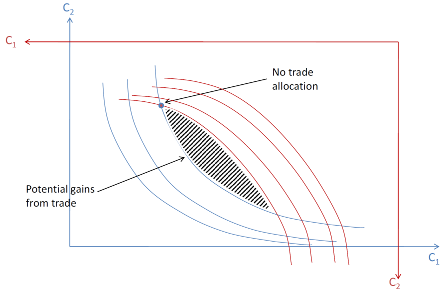
In the above picture, we observe the following.

- The no-line risks are where the consumption for a given agent is equal in both states.
- The lengths of the axis are $w_A^1 + w_B^1$ and $w_A^2 + w_B^2$ respectively.

In this course, we assume that agents $i = A, B$ aim at maximizing a utility function $U_i : \mathbb{R}_+^2 \to \mathbb{R}$ such that

$$U_i(c_i^1, c_i^2) = \pi_1 u_i(c_i^1) + \pi_2 u_i(c_i^2)$$

where π_1 and π_2 are respectively the probability of occurrence of state 1 and 2, $u_i : \mathbb{R}_+ \to \mathbb{R}$ is a function called the utility index. Of course, for each agent, there are multiple c_i^1, c_i^2 with the same value of the utility function (which is, in general, non-injective). As shown in the figure below, from the no-trade situation, there might be consumptions where each agent has higher income compared to the no-trade situation.



Among the possible allocations, particularly interesting are the Pareto-optimal ones. An allocation is Pareto-optimal if and only if there is no alternative feasible outcome at which every individual in the economy is at least as well off and some individual is strictly better off. Graphically, Pareto-optimal allocations are the one where indifferent curves of agents A and B are tangents. Note that there may be many Pareto-optimal allocations, and agents may have different preferences about which is the most interesting one. The contract-curve is the part of the Pareto set for which both agents do at least as well as their initial endowments. This point is very important because both agents gain from the trade, and the outcome is Pareto-optimal.
We now change topic and we dive into the concept of market equilibrium. The setting is that agents, ex-ante, trade assets on the market. The market equilibrium is obtained when asset prices are such that individual strategies are globally compatible, i.e. demand is equal to supply. We denote with Z^i_α (with $\alpha \in \{A, B\}$, $i \in \{1, 2\}$) the quantities of the respective asset bought/ sold by the agent. We assume that ex-ante agents has no wealth. This justifies the constraint $p_1 z^1_i + p_2 z^2_i = 0$. Ex-post, on state j, the agent consumes $w^j_i + z^j_i$. Agent i solves the following:

$$\max_{(z^1_i, z^2_i)} U^i(w^1_i + z^1_i, w^2_i + z^2_i) \text{ subject to } p_1 z^1_i + p_2 z^2_i = 0$$

We say that a market is balanced if the following conditions (known as market clearing conditions) are satisfied.

$$z^1_A + z^1_B = 0$$
$$z^2_A + z^2_B = 0$$

If we can solve the optimization problem for the individual's demand with demands/ supplies that satisfy the market clearing conditions, we have a market equilibrium. If a market equilibrium exists, prices are determined up to a multiplicative scalar. Market equilibrium satisfy the following properties:

- The market equilibrium generates a Pareto-optimal allocation.
- Market equilibrium allocations are comonotone. This is also the case for all Pareto-optimal allocations.

Choice under uncertainty

Consider Z a set of possible consequences. Assume there is a finite set $\Omega = \{1, \ldots, |\Omega|\}$ of possible states of the world and consider $L(Z)$ the set of lotteries with consequences in Z. A lottery is a list of pairs $(x^\omega, \pi_\omega)_{\omega \in \Omega}$ where π_ω gives the probability that the state of the world ω will occur, and $x^\omega \in Z$ is the outcome that is realized when state ω occurs. A compound lottery is a lottery whose prizes are themselves lotteries. The compound lottery with parameter $\lambda \in [0, 1]$ consists in playing the first lottery with probability λ and the second one with proability $1 - \lambda$. The resulting lottery $\lambda \cdot L + (1 - \lambda) \cdot L'$ is called mixture operation.
We want to have a theory of preferences defined over $L(Z)$. A relation of preferences $\succeq$ on $L(Z)$ is a binary relation which is:

- **Complete:** for any $L, L' \in L(Z)$ we have $L \succeq L'$ or $L' \succeq L$.
- **Transitive:** if both $L \succeq L'$ and $L' \succeq L''$, then $L \succeq L''$.
- **Independent:** for all L, L', L'' and $\lambda \in (0, 1)$ then $L \succeq L'$ if and only if $\lambda \cdot L + (1 - \lambda) \cdot L'' \succeq \lambda \cdot L' + (1 - \lambda) \cdot L''$. This is the first axiom of Von-Neumann and Morgenstern.
- **Continuity:** if $L \succeq L' \succeq L''$, then there exists $\lambda \in [0, 1]$ such that $L \sim \lambda \cdot L + (1 - \lambda) \cdot L''$.

A relation of preferences is represented by a utility function $V : L(Z) \to \mathbb{R}$ if and only if

$$L \succeq L' \Leftrightarrow V(L) \geq V(L')$$

Theorem (von Neumann-Morgenstern) If $\succeq$ is a preference relation on $L(Z)$ that satisfies the previous axioms, then there exists $u : Z \to \mathbb{R}$ such that $(x^\omega, \pi_\omega) \succeq (\tilde{x}^\omega, \tilde{\pi}_\omega)$ if and only if $\sum_\omega \pi_\omega u(x^\omega) \geq \sum_\omega \tilde{\pi}_\omega u(\tilde{x}^\omega)$ which can also be rewritten as

$$\mathbb{E}_{\pi_\omega}[u(x^\omega)] \geq \mathbb{E}_{\tilde{\pi}_\omega}[u(\tilde{x}^\omega)]$$

We denote the average of a lottery L by $\mathbb{E}_L[x] = \sum_{\omega \in \Omega} \pi_\omega x^\omega$. Let $\delta_{\mathbb{E}_L[x]}$ the degenerate lottery that gives the expected pay-off of L with probability one. An agent is said to be (weakly) risk averse if and only if for all $L \in L(Z)$

$$\delta_{\mathbb{E}_L[x]} \succeq L$$

This happens if and only if u is concave.
We say that an agent dislikes increases in risk if and only if her utility index u is concave. Different possibilities to define increases in risk are:

- L is riskier than L' if L' is SOSD (i.e. L' second order stochastically dominates L).
- L is riskier than L' if L is a mean preserving spread of L'.
- L is riskier than L' if $L = L' + \varepsilon$ where $\mathbb{E}[\varepsilon] = 0$.

We define a white noise any random variable ε such that $\mathbb{E}[\varepsilon | L = y] = 0$ holds for all y. A risk averse agent will always prefer a lottery L to $L + \varepsilon$.

For any lottery L, its **certainty equivalent** is the amount $e_L \in \mathbb{R}$ such that $\delta_{e_L} \sim L$. In other words, e_L is the amount of money for which the individual is indifferent between the gamble L and a fixed amount of money. Formally, for a VNM agent with utility index u, we have $u(e_L) = \mathbb{E}_L[u(x)]$. Note that for a risk averse agent $\mathbb{E}_L[x] - e_L > 0$, while for a risk neutral agent $\mathbb{E}_L[x] = e_L$.

For any L, the **risk premium** is $\pi_L = \mathbb{E}_L[x] - e_L$. Of course, we have $u(\mathbb{E}_L[x] - \pi_L) = \mathbb{E}_L[u(x)]$. A risk neutral agent associates risk zero to any lottery L, while a (strictly) averse agent a positive risk to any non-degenerate lottery.

Consider $x \in \mathbb{R}$ and the lottery $L = x(1+\varepsilon)$, where ε is a small noise with $\mathbb{E}[\varepsilon] = 0$. We define:

- The relative risk premium $\pi_r(x)$ as
$$\pi_r(x) = -\frac{1}{2}\frac{xu''(x)}{u'(x)}\mathbb{E}\left[\varepsilon^2\right]$$
- The relative risk aversion coefficient $R_r(x) = -x\frac{u''(x)}{u'(x)}$. So we have $\pi_r(x) = -\frac{1}{2}R_r(x)\mathbb{E}\left[\varepsilon^2\right]$.

Some notable examples are:

- If $u(x) = \frac{1-e^{-kx}}{k}$, then $R_a(x) = k$ and $R_r(x) = kx$.
- If $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, then $R_a(x) = \frac{\gamma}{x}$ and $R_r(x) = \gamma$.
- If $u(x) = \log(x)$, then $R_a(x) = \frac{1}{x}$ and $R_r(x) = 1$.

Consider agents A and B with VNM preferences represented by smooth utility indices u_A and u_B. It is equivalent to say:

1. For any lottery L the certainty equivalent defined by A's preferences is smaller than or equal to the one defined by B's.
2. For any lottery L the risk premium defined by B's preferences is smaller than or equal to the one defined by A's.
3. u_A is more concave than u_B. There exists an increasing concave function ϕ such that $u_A = \phi(u_B)$.
4. For any x we have
$$R_a^A(x) = -\frac{u_A''(x)}{u_A'(x)} \geq -\frac{u_B''(x)}{u_B'(x)} = R_a^B(x)$$

It is then said that A is at least as risk averse as B.

Demand for risk

Consider an agent with wealth W who can invest at time zero in a risk-free asset with return r_0 and in a risky asset with random return $\tilde{r}$. If the agent invests an amount α in the risky asset, he gets in period 1:

$$(W-\alpha)r_0 + \alpha \cdot \tilde{r} = Wr_0 + \alpha(\tilde{r} - r_0)$$

With $\omega = Wr_0$ and $\tilde{R} = \tilde{r} - r_0$ the agent problem rewrites

$$\max_a V(\alpha) = \mathbb{E}\left[u(\omega + \alpha\tilde{R})\right]$$

When is the demand for risky asset positive? Let's start deriving some properties of $V(\alpha)$:

$$V'(\alpha) = \mathbb{E}\left[\tilde{R}u'(\omega + \alpha\tilde{R})\right]$$
$$V''(\alpha) = \mathbb{E}\left[\tilde{R}^2u''(\omega + \alpha\tilde{R})\right]$$

Thus, if the agent is risk averse, V is concave. V' is therefore non-increasing. This gives two results:

- If $\mathbb{E}\left[\tilde{R}\right] > 0$, the agent should purchase some risky asset (no matter how risk averse it might be).
- If $\mathbb{E}[\tilde{r}] \leq r_0$, then it is never optimal to hold a positive quantity of the risky asset.

To see this, consider $V'(0) = \mathbb{E}\left[\tilde{R}u'(\omega)\right] = \mathbb{E}\left[\tilde{R}\right]u'(\omega)$. But how much should the agent invest? The first order condition $V'(\alpha^*)$ gives

$$\mathbb{E}\left[\tilde{R}u'(\omega + \alpha^*\tilde{R})\right] = 0$$

which has solution under specific assumptions on $\tilde{R}$ and u. For example, if $\tilde{R}$ is always of the same sign, there is no interior solution. And even if the agent is risk averse and $\tilde{R}$ changes sign, there may not be an interior solution. In the following, we assume that an interior solution exists. Moreover, we will assume that $\mathbb{E}\left[\tilde{R}\right] > 0$ so that the solution is positive (i.e. $\alpha^* > 0$).

Small risks: assume $\tilde{R} = \varepsilon + \tilde{R}'$ with $\mathbb{E}\left[\tilde{R}'\right] = 0$ and consider a small $\varepsilon > 0$. Denote by $\alpha(\varepsilon)$ the solution to

$$\max_\alpha \mathbb{E}\left[u(\omega + (\varepsilon + \tilde{R}')\alpha)\right]$$

We have FOC:

$$\mathbb{E}\left[(\varepsilon + \tilde{R}')u'(\omega + (\varepsilon + \tilde{R}')\alpha(\varepsilon))\right] = 0$$

We know $\alpha(0) = 0$. Then using a Taylor expansion we have $\alpha(\varepsilon) \simeq \alpha(0) + \alpha'(0)\varepsilon = \alpha'(0)\varepsilon$ and

$$\begin{aligned} u'(\omega + (\varepsilon + \tilde{R}')\alpha(\varepsilon)) &= u'(\omega + \alpha(\varepsilon)\tilde{R}) \\ &\simeq u'(\omega) + u''(\omega)\alpha(\varepsilon)\tilde{R} \\ &\simeq u'(\omega) + u''(\omega)\varepsilon\alpha'(0)\tilde{R} \end{aligned}$$

The FOC gives

$$\begin{aligned} &\mathbb{E}\left[(\varepsilon + \tilde{R}')(u'(\omega) + \varepsilon\alpha'(0)\tilde{R}u''(\omega))\right] \\ &= \varepsilon u'(\omega) + \varepsilon^2\alpha'(0)\mathbb{E}\left[\tilde{R}\right]u''(\omega) + \varepsilon\alpha'(0)\mathbb{E}\left[\tilde{R}\tilde{R}'\right]u''(\omega) \\ &= \varepsilon u'(\omega) + \varepsilon\alpha'(0)\mathbb{E}\left[\tilde{R}'^2\right]u''(\omega) = 0 \end{aligned}$$

that is

$$\alpha'(0) = \frac{1}{\mathbb{E}\left[\tilde{R}'^2\right]} \cdot \frac{1}{R_a(\omega)}$$

and

$$\alpha \simeq \varepsilon\alpha'(0) = \frac{\mathbb{E}\left[\tilde{R}\right]}{Var\left[\tilde{R}\right]}\frac{1}{R_a(\omega)}$$

The share of wealth invested in the risky asset is

$$\frac{\alpha}{\omega} \simeq \frac{\mathbb{E}\left[\tilde{R}\right]}{Var\left[\tilde{R}\right]}\frac{1}{R_r(\omega)}$$

Quadratic utility indices: assume $u(c) = c - \frac{\beta}{2}c^2$ with $c = \omega + \alpha\tilde{R}$. Then the FOC writes

$$(1-\beta\omega)\mathbb{E}\left[\tilde{R}\right] - \alpha\beta\mathbb{E}\left[\tilde{R}^2\right] = 0$$

So

$$\alpha = \frac{1-\beta\omega}{\beta}\frac{\mathbb{E}\left[\tilde{R}\right]}{\mathbb{E}\left[\tilde{R}^2\right]}$$

Thus, the amount invested in the risky asset (α) only depends on the mean and the variance of the difference between the return to the risky and the safe asset ($\tilde{R} = \tilde{r} - r_0$).

Hyperbolic absolute risk aversion: consider the HARA utility index with $c = \omega + \alpha\tilde{R}$ and $u(c) = \frac{\gamma}{1-\gamma}(b + \frac{c}{\gamma})^{1-\gamma}$. Combining $u'(c)$ with the FOC gives

$$\mathbb{E}\left[\tilde{R}(\omega + \alpha\tilde{R} + b\gamma)^{-\gamma}\right] = 0$$
$$\Leftrightarrow$$
$$\mathbb{E}\left[\tilde{R}(1 + \frac{\alpha}{(\frac{\omega}{\gamma} + b)}\frac{\tilde{R}}{\gamma})^{-\gamma}\right] = 0$$

Denote by a the solution of $\mathbb{E}\left[\tilde{R}(1 + a\frac{\tilde{R}}{\gamma})^{-\gamma}\right] = 0$. We have

$$a = \frac{\alpha}{\frac{\omega}{\gamma} + b} \Rightarrow \alpha = a(\frac{\omega}{\gamma} + b)$$

Two important special cases of HARA are:

- CARA ($\gamma \to \infty$): α is independent of ω.
- CRRA ($b = 0$): α is proportional to ω.

We now show a few important results. First, we have that **the demand for the risky asset decreases with risk aversion**.

To see this, consider $u_1 = \phi(u_2)$ with ϕ concave. Assume that α_1 and α_2 solve

$$\mathbb{E}\left[\tilde{R}u_1'(\omega + \alpha_1\tilde{R})\right] = 0$$
$$\mathbb{E}\left[\tilde{R}u_2'(\omega + \alpha_2\tilde{R})\right] = 0$$

i.e. $V_1'(\alpha_1) = v_2'(\alpha_2) = 0$. The concavity of ϕ implies:

- If $R \geq 0$ we have $u_2(\omega + \alpha R) \geq u_2(\omega)$ which implies $R\phi'(u_2(\omega + \alpha R)) \leq R\phi'(u_2(\omega))$.
- If $R < 0$ we have $u_2(\omega + \alpha R) < u_2(\omega)$ which implies $R\phi'(u_2(\omega + \alpha R)) \leq R\phi'(u_2(\omega))$.

Wrapping up, $\forall\alpha \in [0,1]$ and $\forall R \in \mathbb{R}$

$$R\phi'(u_2(\omega + \alpha R)) \leq R\phi'(u_2(\omega))$$

Therefore,

$$\begin{aligned} V_1'(\alpha_2) &= \mathbb{E}\left[\tilde{R}u_1'(\omega + \alpha_2\tilde{R})\right] \\ &= \mathbb{E}\left[\tilde{R}u_2'(\omega + \alpha_2\tilde{R})\phi'(u_2(\omega + \alpha\tilde{R}))\right] \\ &\leq \mathbb{E}\left[\tilde{R}u_2'(\omega + \alpha_2\tilde{R})\phi'(u_2(\omega))\right] \\ &= \phi'(u_2(\omega))V_2'(\alpha_2) \\ &= 0 = V_1'(\alpha_1) \end{aligned}$$

And, since V is concave, $\alpha_1 \leq \alpha_2$.

Mean Variance Analysis

We assume that the preferences over portfolio are increasing wrt expected return and decreasing wrt variance. In other words, given returns, we minimize the variance. Reciprocally, given variance, the return is maximized. This leaves one parameter free, which is interpreted as the agents' risk aversion parameter, which determines their choice upon the risk/ reward trade-off.
The general setting for this chapter is the following. We consider a probability space Ω, where $\omega \in \Omega$ is a state of the world. We denote by $\tilde{a}_k = (a_k^\omega)_{\omega\in\Omega}$ the random payoff of asset k and by p_k its price. Denote:

- $\tilde{R}_k = (R_k^\omega)_{\omega\in\Omega}$ the gross return of asset k: $R_k^\omega = \frac{a_k^\omega}{p_k}$. We have: $\tilde{R}_k \in \mathbb{R}^{|\Omega|\times 1}$.
- R_k is a real number defined as $\mathbb{E}\left[\tilde{R}_k\right]$.
- $R = (R_1, \dots, R_k)^T \in \mathbb{R}^{K\times 1}$ and $\tilde{R} = (\tilde{R}_1, \dots, \tilde{R}_k)^T \in \mathbb{R}^{|\Omega|\times K}$.
- The matrix $\Sigma \in \mathbb{R}^{K\times K}$ is the covariance matrix of asset returns. We denote with σ_{kl} the entry of Σ on the k-th row and l-th column.

We have:

$$\mathbb{E}_p\left[\tilde{R}_p\right] = x^T R$$
$$Var_\pi\left[\tilde{R}_p\right] = \sum_{k,l}\sigma_{kl}x_k x_l = x^T\Sigma x$$

Minimum variance portfolio with two stocks: given $R_1, R_2, \sigma_1^2, \sigma_2^2, \rho_{12}$, how does the investor chose x_1 to minimize the variance of the portfolio? The optimization problem to be solved is the following: minimize

$$\frac{1}{2}\sigma_p^2 = \min x_1^2\sigma_1^2 + 2x_1x_2\sigma_1\sigma_2\rho_{12} + x_2^2\sigma_2^2$$

s.t. $x_1 + x_2 = 1$. By setting the derivative to zero we get:

$$x_1^{MVP} = \frac{\sigma_2^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho 12}$$
$$x_2^{MVP} = \frac{\sigma_1^2 - \sigma_1\sigma_2\rho_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho 12}$$
$$\sigma_p^{2,MVP} = \frac{\sigma_1^2\sigma_2^2(1-\rho_{12}^2)}{\sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho_{12}}$$

Note that:

- With K assets, identical variances ($\sigma_1^2 = \dots = \sigma_K^2 = 0$) and independence ($\rho_{jk} = 0\forall j \neq k$), the optimal minimum variance portfolio composition is such that the same amount is divided in each stock and we get $\sigma_p^{2,MVP} = \frac{\sigma_1^2}{k}$. This is the idea of diversification.
- With perfectly negative correlation ($\rho_{jk} = -1\forall j \neq k$), we can obtain a portfolio with no risk.

In the general case, the problem of the investor can be written as

$$\frac{1}{2}\sigma^2 = \min \frac{1}{2}\sum_{k,l}\sigma_{kl}x_kx_l s.t.$$
$$\sum_k x_kR_k = m$$
$$\sum_k x_k = 1$$

or, in matrix form

$$\frac{1}{2}\sigma^2 = \min \frac{1}{2}x^T\Sigma x s.t.$$
$$x^TR = m$$
$$x^T\vec{1} = 1$$

Using Lagrange multipliers and defining $\Sigma^{-1} = \Gamma = (\gamma^{kl})_{(k,l)\in\{1,\ldots,K\}^2}$ we get:

$$x_k = \lambda\sum_l \gamma^{kl}R_l + \mu\sum_l\gamma^{kl}(\text{i.e } x = \lambda\Gamma R + \mu\Gamma\vec{1})$$
$$c := R^T\Gamma R$$
$$b := R^T\Gamma\vec{1}$$
$$d := \vec{1}^T\Gamma\vec{1}$$
$$\Delta := dc - b^2$$
$$\lambda = \frac{dm - b}{\Delta}$$
$$\mu = \frac{-bm + c}{\Delta}$$
$$\sigma^2 = \lambda m + \mu = \frac{dm^2 - 2bm + c}{\Delta}$$
$$(\frac{\Delta}{d})\sigma^2 - (m - \frac{b}{d})^2 = \frac{\Delta}{d^2}$$

The **global minimum variance portfolio** is the portfolio $x_G \in \mathbb{R}^K$ with smallest variance. We get:

$$x_{G,k} = \frac{1}{d}\sum_l \gamma^{kl}$$
$$m_G = \frac{b}{d}$$
$$\sigma_G^2 = \frac{1}{d}$$

The **mutual fund theorem** says that given two frontier portfolios x_A and x_B, all efficient portfolios can be obtained by mixing x_A and x_B in different proportions. The linear combination of efficient portfolios is always efficient.

Capital Asset Pricing Model

We now extend the framework of the previous section to the case of a portfolio comprising one safe asset x_0 and K risky assets $r_{-0} = (x_1, \ldots, x_K)^T$. The investor's program is:

$$\frac{1}{2}\sigma^2 = \min \frac{1}{2}\sum_{k,l}\sigma_{kl}x_kx_l$$
$$x_0R_0 + \sum_k x_kR_k = m$$
$$x_0 + \sum_k x_k = 1$$

By using the FOC we get the result $x_k = \lambda\sum_l\gamma^{kl}(R_l - R_0)$ (i.e. $x_{-0} = \lambda\Gamma(R - R_0\vec{1})$) and $x_0 = 1 - \sum_k x_k$. Therefore, we see that for any efficient portfolio, the sub-portfolio of risky assets x_{-0} is proportional to $\sum_l\gamma^{kl}(R_l - R_0)$.
We define the **tangent portfolio** x_T as the only efficient portfolio that does not use the safe asset. So we have $x_{T,0} = 0$. By using the previous result we also obtain $\forall k \in \{1, \ldots, K\}$

$$x_{T,k} = \frac{\sum_l\gamma^{kl}(R_l - R_0)}{\sum_{j,l}\gamma^{jl}(R_l - R_0)}$$

Moreover, we have

$$x_k = \lambda\sum_{j,l}\gamma^{jl}(R_l - R_0)x_{T,k}$$

that is,

$$x_{-0} = \lambda\sum_{j,l}\gamma^{jl}(R_l - R_0)x_T$$

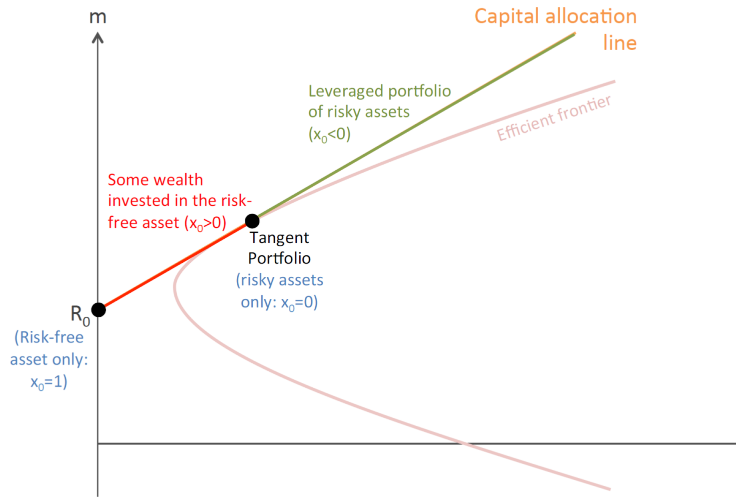
Hence, the investor's holdings of risky assets are proportional to this tangent portfolio. Therefore, for any efficient portfolio, we have

$$x = \alpha y_0 + (1 - \alpha)x_T$$

where $y_0^T = (1, 0, \ldots, 0)$ and $x_T^T = (0, x_{T,1}, \ldots, x_{T,K})$ is the tangent portfolio. How do we choose α? We have:

$$m = \alpha\cdot R_0 + (1-\alpha)R_T = (1-\alpha)(R_T - R_0) + R_0$$
$$\sigma = (1-\alpha)\sigma_T$$
$$\implies m = (\frac{R_T - R_0}{\sigma_T})\sigma + R_0$$

Mutual fund theorem II: In the presence of a safe asset, efficient portfolios are a combination of the risk-free asset and the tangent portfolio. The exact combination is determined by m.
Corollary: All efficient portfolios are located on a real line in the (σ, m) plane; this line is called capital allocation line.



For any portfolio with expected return m and standard deviation σ, we define the Sharp ration as

$$\frac{m - R_0}{\sigma}$$

We now introduce the Capital Asset Pricing Model (CAPM) which adds two key assumptions to the Markovitz model:

- Complete agreement: all agents agree on both m and Σ.
- Borrowing and lending at a risk-free rate is available to all investors.

Under these assumptions, all investors see the same opportunity set and hold the same tangent portfolio of risky assets, which must therefore be the value-weight market portfolio. The CAPM rests on the notion of market portfolio, which is simply the aggregation of the economy's financial assets. The capital allocation line is known to all investors, as it can be read in the market.

Theorem (CAPM): at equilibrium, the expected return of asset k satisfies

$$R_k - R_0 = \underbrace{\frac{Cov_\pi(\tilde{R}_k, \tilde{R}_M)}{Var(\tilde{R}_M)}}_{\beta_k}(R_M - R_0)$$

In words, the risk premium associated with asset k is given by the product of the asset's β with the risk premium on the market portfolio.

Derivation of the theorem: from the mutual fund theorem we know that for any efficient portfolio $\sum_k \sigma_{kl} x_k = \lambda(R_l - R_0)$. We now define the market portfolio $z_M = (z_{M,0}, \dots, z_{M,k})$ as the portfolio consisting of all assets held in the economy. For $k = (0, \dots, K)$, the share of wealth held in asset k is given by

$$x_{M,k} = \frac{p_k z_{M,k}}{\sum_\kappa p_\kappa z_{M,\kappa}}$$

Since the market is assumed to be efficient, we have

$$\sum_k \sigma_{kl} x_{M,k} = \lambda(R_l - R_0)$$

Moreover we have

$$\tilde{R}_M = \sum_\kappa x_{M,\kappa} \tilde{R}_\kappa$$

$$Cov_\pi(\tilde{R}_l, \tilde{R}_M) = \sum_k x_{M,k} Cov_\pi(\tilde{R}_k, \tilde{R}_l)$$

$$= \sum_k \sigma_{kl} x_{M,k}$$

Hence, using $\sum_k \sigma_{kl} x_{M,k} = \lambda(R_l - R_0)$ we obtain $Cov_\pi(\tilde{R}_l, \tilde{R}_M) = \lambda(R_l - R_0)$. Multiplying both sides by $x_{M,l}$ and summing, one gets

$$\sum_l x_{M,l} Cov_\pi(\tilde{R}_l, \tilde{R}_M) = \lambda(R_M - R_0)$$

or also

$$Var(\tilde{R}_M) = \lambda(R_M - R_0)$$

Remarks:

- We call idiosyncratic a risk which has zero correlation with the market. According to the CAPM relation, an idiosyncratic risk is worth zero premium. This does not mean that the risk is liked by all investors, it just means that there is zero net supply or demand in the economy for that risk. Note that at equilibrium, no agent bears any idiosyncratic risk, as every agent then detains a combination of the market portfolio and the safe asset.
- We call a systematic risk a risk which has correlation one to the market risk and which can not be avoided when investing in the market.
  - The CAPM relations show that, for the same expected values, assets which are negatively correlated with the market have a greater price (and, therefore, a lower expected return). Indeed there is a net demand for those assets in the economy, as they insure their holders against systematic risk.
  - Conversely, assets which are positively correlated with the market are in net supply: people dislike them as they increase their systematic risk, thus ask for a higher risk premium to carry them.

Risk Sharing and Insurance

We start by presenting the setting for this section. We have a single good and two dates. At date $t = 0$ the contingent exchange contracts are signed. At date $t = 1$ the contracts are settled. Moreover:

- We consider a probability space Ω. $\omega \in \Omega$ is a state of the world.
- π_ω with $\sum_\omega \pi_\omega = 1$ is the known probability distribution on Ω.
- If ω occurs, a total amount of e^ω resources is shared across agents (here e is just a notation, it does not have anything to do with the natural number).
- There is a finite number of agents with VNM preferences and are risk averse. The utility index of agent i is denoted u_i and satisfies the Inada condition

$$\lim_{c \to 0} u_i'(c) = \infty$$

Definition: an allocation $c = (c_i^\omega)_{1 \le i \le l, \omega \in \Omega}$ with $c_i^\omega \ge 0$ is the specification of a contingent assumption plan for all the individuals i in each state of the world.
Definition: an allocation is feasible if in each state ω, total consumption equals total resources, i.e.

$$\forall \omega \sum_{i=1}^{l} c_i^\omega = e^\omega$$

Definition: an allocation is Pareto optimal if it is feasible and if there is no other feasible allocation $\tilde{c}$ such that $\forall i \in \{1, \dots, l\}$

$$\mathbb{E}_\pi [u_i(\tilde{c}_i)] \ge \mathbb{E}_\pi [u_i(c_i)]$$

with at least one strict inequality.

Proposition (Borch): if all agents are strictly risk averse, and have identical beliefs, then for any Pareto-optimal allocation,

$$\forall i \in \{1, \dots, l\}, e^\omega = e^{\omega'} \Rightarrow c_i^\omega = c_i^{\omega'}$$

A feasible allocation c such that $c_i^\omega > 0$ is Pareto optimal if and only if there are nonnegative weights $(\lambda^i)_{1\leq i\leq l}$ and $(\mu^\omega)_{\omega\in\Omega}$ such that $\forall\omega\in\Omega$ and $\forall i\in\{1,\dots,l\}$

$$\lambda^i u_i'(c_i^\omega) = \mu^\omega$$

This condition can be checked as follows:

$$\forall i\neq j\,\frac{u_i'(c_i^\omega)}{u_j'(c_j^\omega)} = \frac{\lambda_j}{\lambda_i}$$
$$\forall \omega\neq\omega'\,\frac{u_i'(c_i^\omega)}{u_i'(c_i^{\omega'})} = \frac{\mu^\omega}{\mu^{\omega'}}$$

Corollary 1: assuming that all agents are risk averse, for any Pareto-optimal allocation the c_i^ω and e^ω are comonotonic, i.e.

$$\forall i\forall(\omega,\omega')\in\Omega^2 e^\omega\geq e^{\omega'}\Leftrightarrow c_i^\omega\geq c_i^{\omega'}$$

Corollary 2: assuming that there is a risk-neutral agent and that the allocation $(c_i)_{1\leq i\leq l}$ is Pareto-optimal, we have

$$\forall(\omega,\omega')\in\Omega^2 c_i^\omega = c_i^{\omega'}$$

Risk Sharing and Asset Pricing in a Market Equilibrium

We start with the setting of the chapter.

- Ω is a probability space. $\omega\in\Omega$ is a state of the world.
- We have I agents living in two periods.
- Endowments for each agent i.
  - At time $t=0$, deterministic endowment e_i^0.
  - At time $t=1$, random endowments $\tilde{e}_i = (e_i^\omega)_{\omega\in\Omega}$.
- Consumption for each agent i.
  - At $t=0$, c_i^0.
  - At $t=1$, $\tilde{c}_i = (c_i^\omega)_{\omega\in\Omega}$.
- Let's denote $\tilde{C}_i = (c_i^0,\tilde{c}_i)$.

The goal is maximizing

$$U_i(\tilde{C}_i) = u_i(c_i^0) + \delta\mathbb{E}_\pi\left[v_i(\tilde{c}_i)\right]$$

where we assume $v_i(c) = c-\alpha_i c^2$. In the market there is a risk free asset and K risky assets. This introduces the following budget constraints for all ω:

$$c_i^0 = e_i^0 - \frac{z_i^0}{1+r} - \sum_{k=1}^K p_k z_i^k$$
$$c_i^\omega = e_i^\omega + z_i^0 + \sum_{k=1}^K z_i^k a_k^\omega$$

The FOC for the optimization problem yield:

$$u_i'(c_i^0) = (1+r)\delta\mathbb{E}\left[v_i'(\tilde{c}_i)\right]$$
$$p_k u_i'(c_i^0) = \delta\mathbb{E}\left[\tilde{a}_k v_i'(\tilde{c}_i)\right]$$

which can be combined to

$$\begin{aligned}(1+r)p_k\mathbb{E}\left[v_i'(\tilde{c}_i)\right] &= \mathbb{E}\left[\tilde{a}_k v_i'(\tilde{c}_i)\right]\\ &= \mathbb{E}\left[\tilde{a}_k\right]\mathbb{E}\left[v_i'(\tilde{c}_i)\right] + Cov(\tilde{a}_k, v_i'(\tilde{c}_i))\end{aligned}$$

Using $v_i'(c) = 1-2\alpha_i c$ we get $Cov(\tilde{a}_k, v_i'(\tilde{c}_i)) = -2\alpha_i Cov(\tilde{a}_k,\tilde{c}_i)$ and

$$\frac{1}{2\alpha_i}\mathbb{E}\left[v_i'(\tilde{c}_i)\right]\left(\mathbb{E}\left[\tilde{a}_k\right]-(1+r)p_k\right) = Cov(\tilde{a}_k,\tilde{c}_i)$$

Using $\tilde{c}_i = \tilde{e}_i + z_i^0 + z_i^T\tilde{a}$ and denoting by Σ the covariance matrix of a, we get

$$\Sigma\cdot z_i = -Cov(\tilde{a},\tilde{e}_i) + \frac{1}{2\alpha_i}\mathbb{E}\left[v_i'(\tilde{c}_i)\right]\left(\mathbb{E}\left[\tilde{a}\right]-(1+r)p_k\right)$$

Define by T_i^a the absolute risk tolerance of investor i:

$$T_i^a(c) = -\frac{v_i'(c)}{v_i''(c)}$$

Moreover, define the aggregate risk tolerance

$$T^a(c_M) = \sum i = 1^l T_i^a(c_i)$$

Since in the case of $v_i'(c)$ is linear, we have $\mathbb{E}\left[v_i'(\tilde{c}_i)\right] = v_i'(\mathbb{E}\left[\tilde{c}_i\right])$ and $v_i''(c) = -2\alpha_i$. Therefore the formula above becomes

$$\Sigma\cdot z_i = -Cov(\tilde{a},\tilde{e}_i) + T_i^a(\mathbb{E}\left[\tilde{c}_i\right])\left(\mathbb{E}\left[\tilde{a}\right]-(1+r)p\right)$$

Note that $\sum_{i=1}^l z_i = 0$, which implies $\sum_{i=1}^l c_i^\omega =: c_M^\omega = e_M^\omega := \sum_{i=1}^l e_i^\omega$. We get:

$$\Sigma\left(\sum_{i=1}^l z_i\right) = \sum_{i=1}^l\left(T_i^a(\mathbb{E}\left[\tilde{c}_i\right])\left(\mathbb{E}\left[\tilde{a}\right]-(1+r)p\right) - Cov(\tilde{a},\tilde{e}_i)\right)$$

which implies (exploiting that for quadratic utility $\sum_{i=1}^l T_i^a(\mathbb{E}\left[\tilde{c}_i\right]) = T^a(\mathbb{E}\left[\tilde{e}_M\right])$)

$$0 = T_i^a(\mathbb{E}\left[\tilde{e}_M\right])\left(\mathbb{E}\left[\tilde{a}\right]-(1+r)p\right) - Cov(\tilde{a},\tilde{e}_M)$$

We hence derived the following fundamental result.

The consumption base CAPM: for every asset $k = 1,\dots,K$

$$p_k = \frac{1}{1+r}\left(\mathbb{E}\left[\tilde{a}_k\right] - \frac{Cov(\tilde{e}_M,\tilde{a}_k)}{T^a(\mathbb{E}\left[\tilde{e}_M\right])}\right)$$